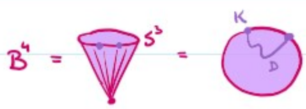
Concordance Lecture 2

September 9, 2021 1:35 PM

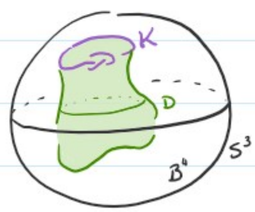
Recall A knot in S^3 is trivial iff it bounds an emb. disk in S^3

HW Every knot in S^3 bounds an emb. disk in S^4

adds 1 dimension, but what about ½ a dimension?



Defn A knot $K \subseteq S^3$ is ~~topologically~~ smoothly **slice** if there exists a ~~topologically~~ smoothly locally flat[1] properly-embedded[2] disk $D \subseteq B^4$ with $K = \partial D$. We call D a **slice disk** for K.



(☆) Generalization of "trivial knots"

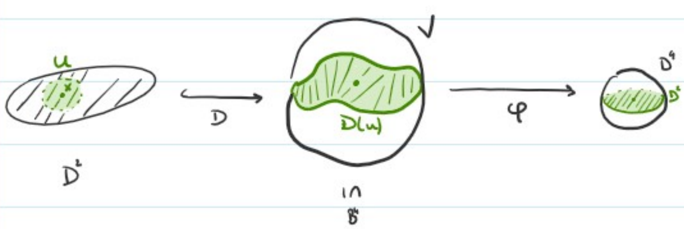
i.e. some nontrivial knots are slice

(?) Are all knots slice? (No)

↳ need obstructions/invariants

(!) Need locally flat (see HW)

An embedding $D: D^2 \hookrightarrow B^4$ is **locally-flat*** if $\forall x \in D^2$ $\exists$ nbhds $\begin{cases} U \subset D^2 \text{ of } x \\ V \subset B^4 \text{ of } D(x) \end{cases}$ such that $(V, D(U)) \overset{\varphi}{\approx} (\mathring{D}^4, \mathring{D}^2)$



* different if $x \in \partial D^2$

Movies visualizing slice disks $D \subset B^4$ by their level sets

view B^4 as $S^3 \times [0,1] / S^3 \times \{0\}$

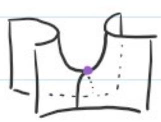
$r : (x,t) \longmapsto t$ gives radius

$\Bigg\} \; L_t := D \cap (S^3 \times \{t\})$

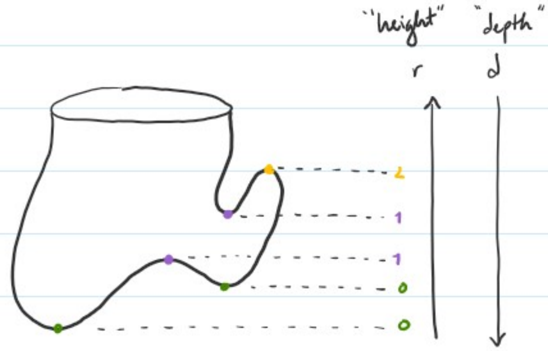
Fact Every slice disk is isotopic (rel ∂) to a disk where r is a Morse function with finitely many isolated critical points, corresponding to:

(0) (local min) birth

(1) saddle



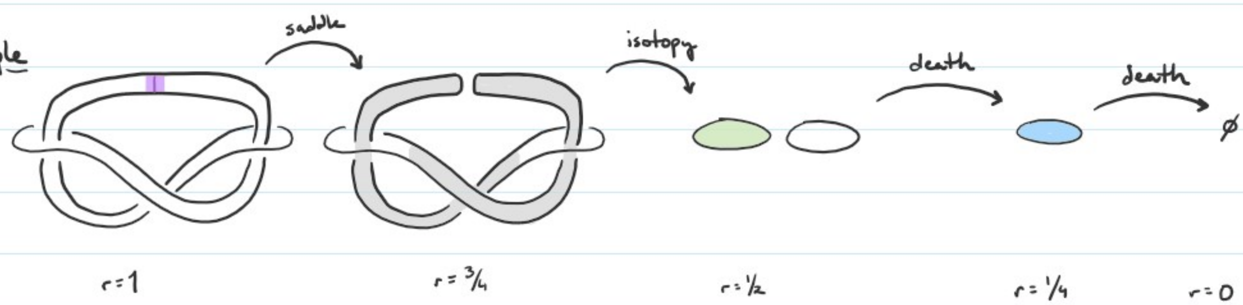
(2) (local max) death



- level sets are generically links, except at levels with a critical pt, where they are links w/ a singularity
- Direction matters: height vs depth have opposite indexed crit. pts.
- Still just a schematic! Level sets are links, not planar S^1's

Defn A *movie* is a finite sequence of level sets $L_1, \dots, L_n$ where L_i and L_{i+1} are related by a birth, death, or saddle.

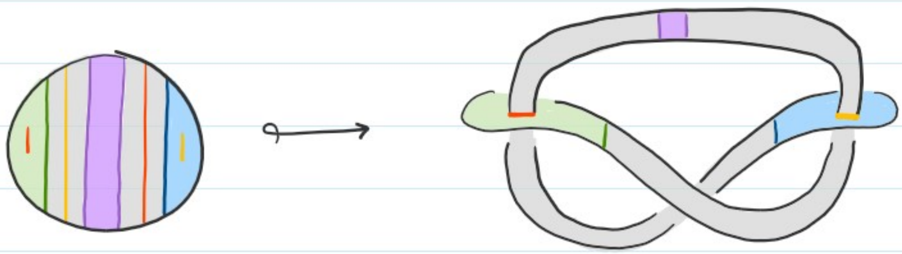
Example



Why is this a disk?

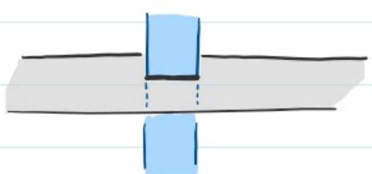
(1) Another schematic: 

(2) Can push D into a single level set (ie $\partial B^4 = S^3$)
to obtain an immersion:



DISK!

In this case, all singularities look like:



Defn This is a ribbon singularity

Defn A knot is ribbon if it bounds a disk in S^3 with only ribbon singularities.

☺ Can make many ribbon knots:

1. Begin w/ unlink
2. Attach bands between consecutive components

(avoid clasp intersections) 

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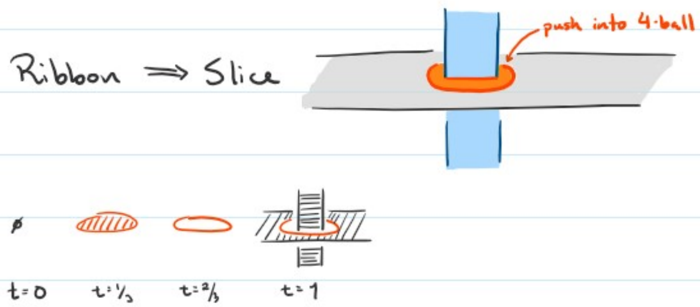
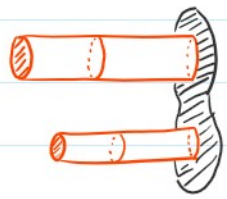
Exercise Ribbon ⟹ Slice





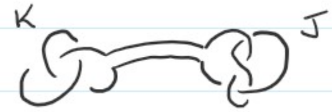
Conjecture (open) (Fox '1960s) Slice ⟹ Ribbon

Exercise A knot is ribbon iff it bounds a smoothly emb. disk in B^4 with only local minima and saddles (wrt radius)

WARNING ∃ slice disks that are not isotopic to a ribbon disk

WARNING Category matters (∃ top slice knots that are not sm. slice)

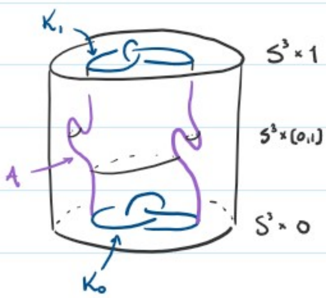
<u>Recall</u> connected sum $K \# J$



$(\{\text{knots}\}, \#)$ is a monoid

$$\boxed{\text{Knots mod slice knots forms a group}}$$

Defn Knots K_0 and K_1 are concordant, $K_0 \sim K_1$, if $\exists$ top $\overset{\text{sm}}{}$ loc-flat annulus $A : S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ with $\partial A = (K_0 \times 0) \sqcup (K_1 \times 1)$



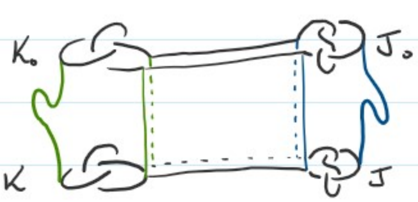
Similarly define movies

<u>Exercise</u> Concordance is an equiv. reln

Defn For a knot K, we call $[K]$ the concordance class of K.
and denote the set of all concordance classes by $\mathcal{C}$ $\quad \mathcal{C}^{TOP}$ or $\overset{\mathcal{C}}{\overset{\|}{\mathcal{C}^{DIFF}}}$

Prop Connected sum descends to a well-defined operation on $\mathcal{C}$
$$[K] + [J] := [K \# J] \quad \text{on } \mathcal{C}.$$

<u>Proof</u> $K_0 \sim K$ and $J_0 \sim J$



Theorem $(\mathcal{C}, \#)$ is a group with identity [unknot] and inverses $-[K] = [-K]$.

see HW: mirror and reverse orientation

Defn The pair $(\mathcal{C}, \#)$ is the **concordance group**.

Thm (HW) $K \sim J$ iff $K \# -J$ is slice.

Cor $\underbrace{K \sim \text{unknot}}_{[K]=0}$ iff K is slice

> "Knots mod concordance forms a group"
> "Knots mod slice knots forms a group"

? We want to construct concordance invariants to study slice knots and concordance